

# A Sufficient Condition for Nilpotency in a Finite Group

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## Abstract

It is shown that finite groups in which the order of the product of every pair of elements of co-prime order is the product of the orders, is nilpotent.

It is well known that every finite nilpotent group  $G$  satisfies the following property.

**Property A.** *The product of any two elements of  $G$  of co-prime orders  $k$  and  $m$  has order  $km$ .*

It does not seem to be known that the converse of this result holds: certainly it does not appear in any standard text that we know. The purpose of this short note is to establish this converse. A suitable reference for all terminology and results used here is [1].

**Theorem.** *Every finite group  $G$  satisfying Property A is nilpotent.*

*Proof.* Note first that Property A extends to more than two elements of mutually co-prime orders, in the obvious way. What we do is to show that every Sylow subgroup is normal, and this is enough to establish nilpotency. Suppose that  $G$  satisfies Property A, let  $p_1, p_2, \dots, p_r$  be the distinct primes dividing the order of  $G$ , and  $S_i$  a Sylow  $p_i$ -subgroup, for each  $i$ . The first step is to establish the equality

$$G = S_1 S_2 \cdots S_r. \quad (1)$$

We do this by counting the number of elements on the right-hand side. Suppose that  $s_i, t_i$  are elements of  $S_i$  for  $i = 1, 2, \dots, r$ , and suppose that we have an equality

$$s_1 s_2 \cdots s_r = t_1 t_2 \cdots t_r.$$

Then

$$s_1 s_2 \cdots s_{r-1} = t_1 t_2 \cdots t_r s_r^{-1}.$$

If  $t_r s_r^{-1}$  is not the identity, then by Property A the element on the right-hand side has order divisible by  $p_r$ , whereas that on the left-hand side does not. This is a contradiction, and so  $s_r = t_r$ . In the same way  $s_i = t_i$  for each  $i$ . Thus, a count of the number of elements in the product  $S_1 S_2 \cdots S_r$  shows that there are as many as there are elements in  $G$ , and this establishes equality (1). Consider now a conjugate  $x$  of an arbitrary element of the Sylow subgroup  $S_1$ . Then by what we just saw,  $x$  is a product  $s_1 s_2 \cdots s_r$  with obvious notation. Since  $x$  is of order a power of  $p_1$ , Property A tells us that  $s_2 = s_3 = \cdots = s_r = 1$  and so that  $x$  is in  $S_1$ . Thus  $S_1$  is normal, and the same goes for all the other  $S_i$ , so all  $S_i$  are normal and  $G$  is nilpotent, as required.  $\square$

**Remark.** We do not know whether every finite group  $G$  has a set of Sylow subgroups, one for each prime dividing the order of  $G$ , such that equality (1) holds. Certainly soluble groups do, because of Philip Hall's celebrated theorem (see [1, p. 665]) stating that every soluble group  $G$  has a set of Sylow subgroups that permute in pairs. Since the Sylow subgroups generate  $G$ , the fact that Sylow subgroups permute in pairs easily yields equation (1).

On the other hand, some insoluble groups have this property. As an easiest example, consider the alternating group  $A_5$  on 1, 2, 3, 4, 5. It has the following Sylow-2, Sylow-3, and Sylow-5 subgroups respectively:  $P = \langle (1, 2)(3, 4), (1, 3)(2, 4) \rangle$ ,  $Q = \langle (1, 2, 3) \rangle$ ,  $R = \langle (1, 2, 3, 4, 5) \rangle$ . Here  $PQ$  is the alternating group on 1, 2, 3, 4, of order 12; since  $R$  is of order 5, we have  $A_5 = PQR$ . Note that the Sylow subgroups have to be chosen carefully: not every choice will do. Other small simple groups have the property under discussion, as M.F. Newman has pointed out in correspondence.

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## References

- [1] Bertram Huppert, *Endliche Gruppen. I* (German), Grundlehren Math. Wiss., **134**, Springer-Verlag, Berlin–New York, 1967.